INTERSECTION NUMBERS ON $\overline{\mathcal{M}}_{g,n}$

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Abstract. We introduce the package `HodgeIntegral` which calculates top intersection numbers among tautological classes on $\overline{\mathcal{M}}_{g,n}$. As an application, we show that the tautological ring of $\mathcal{M}_{3,0}^{\lambda_2}$, or the moduli space of genus three curves whose dual graph has at most one loop, is not Gorenstein.

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of stable curves of genus g with n marked points. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are defined to be the smallest system of $\mathbb{Q}$-subalgebras of the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ that are closed under the natural forgetful morphisms

$$\pi_{n+1}\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \tag{1}$$

and the gluing morphisms

$$\iota_{\mathrm{irr}}\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} \tag{2}$$

$$\iota_{g_1,S}\colon \overline{\mathcal{M}}_{g_1,|S|+1} \times \overline{\mathcal{M}}_{g_2,|S^c|+1} \to \overline{\mathcal{M}}_{g_1+g_2,n}, \tag{3}$$

Here S denotes a subset of $\{1,\dots,n\}$. Tautological rings contain boundary strata, Mumford-Morita κ classes, cotangent ψ-classes, and the chern classes of the Hodge bundle $\lambda_i := c_i(\mathbb{E})$. (For definitions and properties of these tautological classes, see [M, AC].)

Around 1997, Faber implemented the program `KaLa5` in Maple which calculates top intersection numbers among κ, λ and ψ classes [F]. The Macaulay2 package `HodgeIntegral`, written by the author and Greg Smith, is modeled after Faber's program, though the algorithm presented here is slightly different. The main advantage of `HodgeIntegral` over `KaLa5` is that it is entirely recursive; by contrast, `KaLa5` involves the use of look-up tables which limit the calculations to $\dim \overline{\mathcal{M}}_{g,n} \leq 20$. What limits `HodgeIntegral` is, as with all recursions, the need for memory. In practice this poses no problem up to $\dim \overline{\mathcal{M}}_{g,n} \leq 30$, and many cases extend beyond this threshold.

Let $\mathcal{M}_{3,0}^{\lambda_2}$ denote the moduli space of genus three curves whose dual graph has at most one loop; equivalently, this is the locus of curves in $\overline{\mathcal{M}}_{3,0}$ where the sum of the geometric genera of the components is at least 2. We use the package `KaLa5` to show that the tautological ring of $\mathcal{M}_{3,0}^{\lambda_2}$, which is defined by restriction, is not Gorenstein.

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1. Integrals among ψ, κ, and λ classes

Top intersection numbers among ψ classes are determined with the Theorem 1.1 of [LX].

$$\begin{aligned}(2g+n-1)(2g+n-2)\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} = {}& \\ \frac{2d_1+3}{12}\int_{\overline{\mathcal{M}}_{g-1,n+4}} \psi_1^{d_1+1}\psi_2^{d_2}\cdots\psi_n^{d_n} &- \frac{2g+n-1}{6}\int_{\overline{\mathcal{M}}_{g-1,n+3}} \psi_1^{d_1}\cdots\psi_n^{d_n} \\ + \sum_{I\coprod J=\{2,\dots,n\}} (2d_1+3)\int_{\overline{\mathcal{M}}_{g',|I|+3}} \psi_1^{d_1+1}\prod_{i\in I}\psi_i^{d_i} &\int_{\overline{\mathcal{M}}_{g-g',|J|+2}} \prod_{j\in J}\psi_i^{d_i} \\ - \sum_{I\coprod J=\{2,\dots,n\}} (2g-n-1)\int_{\overline{\mathcal{M}}_{g',|I|+2}} \psi_1^{d_1}\prod_{i\in I}\psi_i^{d_i} &\int_{\overline{\mathcal{M}}_{g-g',|J|+2}} \prod_{j\in J}\psi_j^{d_j}\end{aligned} \tag{4}$$

This reduces an integral in ψ classes to a sum of four terms involving integrals on strictly lower genera. Our base cases are:

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \binom{n-3}{d_1, \dots, d_n} \tag{5}$$

Integrals involving both ψ and κ classes can be reduced to the case above using the pullback formulas

$$\pi_{n+1}^* \kappa_b = \kappa_b - \psi_{n+1}^b \tag{6}$$

$$\pi_{n+1}^* \psi_i = \psi_i - D_{i,n+1}, \tag{7}$$

The term $D_{i,n+1}$ is the boundary divisor that is the closure of the locus of curves consisting of a rational component with two marked points p_i and p_{n+1} attached to a genus g curve carrying the remaining marked points. It is immediate from this definition that the product $\psi_{n+1} D_{i,n+1}$ vanishes for all i. These allow us to compute

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \kappa_{b_1} \cdots \kappa_{b_m} = \int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1} \prod_{i=1}^{n} \psi_i^{\alpha_i} \prod_{j=2}^{m} \kappa_{b_j} = \int_{\overline{\mathcal{M}}_{g,n}} (\pi_{n+1*} \psi_{n+1}^{a+1}) \prod_{i=1}^{n} \psi_i^{\alpha_i} \prod_{j=2}^{m} \kappa_{b_j} \tag{8}$$

$$= \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi_{n+1}^{a+1}) \pi_{n+1}^* \left(\prod_{i=1}^{n} \psi_i^{\alpha_i} \prod_{j=2}^{m} \kappa_{b_j} \right) \tag{9}$$

$$= \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a+1} \prod_{i=1}^{n} (\psi_i^{\alpha_i} - D_{i,n+1}) \prod_{j=2}^{m} (\kappa_{b_j} - \psi_{n+1}^{b_j+1}) \tag{10}$$

$$= \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a+1} \prod_{i=1}^{n} \psi_i^{\alpha_i} \prod_{j=2}^{m} (\kappa_{b_j} - \psi_{n+1}^{b_j+1}) \tag{11}$$

The expression on the right can be expanded to a sum of integrals which contain one fewer κ class in their integrands at the cost of introducing a marked point. Repeated iteration of this equation allows us to eliminate κ classes entirely.

Integrals involving λ classes are more complicated. The first step is to express λ classes in terms of the chern character of $\mathbb{E}$ using the formula

$$1 + \lambda_1 t + \cdots + \lambda_g t^g = \exp\left(\sum_{i=1}^{g} (2i-2)! \operatorname{ch}_{2i-1} t^{2i-1} \right) \tag{12}$$

Applying the Grothendieck-Riemann-Roch formula to the universal family $\pi\colon \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,0}$ ([M, Eq. (5.2)]) and pulling this back to $\overline{\mathcal{M}}_{g,n}$ gives us an expression the chern character of $\mathbb{E}$ in terms of κ, ψ, and boundary classes.

$$\operatorname{ch}_a = \frac{B_{a+1}}{(a+1)!} \left(\kappa_a - \sum_{i=1}^{n} \psi_i^a + \frac{1}{2} \sum_{i=0}^{a-1} (-1)^i \sum_{\iota_\beta} (\iota_\beta)_* \, \psi_\star^i \psi_\bullet^{a-1-i} \right). \tag{13}$$

Here B_i denotes the i-th Bernoulli number, and ι_β ranges over all possible gluing morphisms.

$R^*(\mathcal{M}_{3,0}^{\lambda_2})$ IS NOT GORENSTEIN

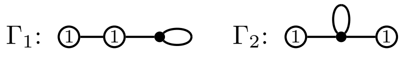
The tautological ring $R^*(\mathcal{M}_{3,0}^{\lambda_2})$ is one-dimensional in degree 4 and vanishes in higher degree [CY, Proposition 1], thus we have an intersection pairing

$$R^i(\mathcal{M}_{3,0}^{\lambda_2}) \times R^{4-i}(\mathcal{M}_{3,0}^{\lambda_2}) \to R^4(\mathcal{M}_{3,0}^{\lambda_2} \cong \mathbb{Q} \tag{14}$$

The chern class λ_2 does not vanish on the generator of $R^4(\mathcal{M}_{3,0}^{\lambda_2})$ and serves as an evaluation class for the pairing.

We use `HodgeIntegral` to show that this pairing is degenerate. Let Γ_1 and Γ_2 denote two dual graphs below:

(15)



and let X_1 and X_2 denote the associated boundary strata. It is straightforward to check that

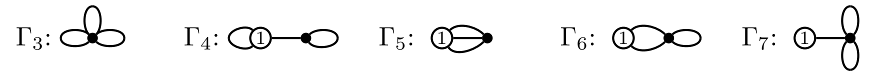
$$(X_1 - X_2)\lambda_2 = 0. \tag{16}$$

We now show that X_1 and X_2 are not algebraically equivalent in $\mathcal{M}_{3,0}^{\lambda_2}$. This is equivalent to checking if their extensions to $\overline{\mathcal{M}}_{3,0}$ satisfy

$$X_1 - X_2 \in R^1(\overline{\mathcal{M}}_{3,0} \setminus \mathcal{M}_{3,0}^{\lambda_2}) \tag{17}$$

The generators of $R^3(\overline{\mathcal{M}}_{3,0} \setminus \mathcal{M}_{3,0}^{\lambda_2})$ are the boundary strata X_i associated to the the respective graphs:

(18)



The intersection pairing of $X_1, \ldots, X_7$ against the five tautological classes κ_3, $\kappa_1\kappa_2$, κ_1^3, $\kappa_2\lambda_2$, and $\kappa_1^2\lambda_1$ is computed with the code below:

```
List * List := (A,B) -> apply(A,B,(x,y)->x*y)
tempFactors = (FactorList,n) -> (
     if #FactorList===0 then return {splice{n:1}}
     else (
          tempList := tempFactors(drop(FactorList,1),n);
          a := first FactorList;
          newList := new List;
               for i from 1 to n do (
               aList = splice{i-1:1,a,n-i:1};
               newList = append(newList, apply(tempList, x->aList*x)));
          return flatten newList));
gnList = {{(1,1),(1,2),(0,3)},{(1,1),(1,1),(0,4)},{(0,6)},{(1,3),(0,3)},
     {(1,3),(0,3)},{(1,2),(0,4)},{(1,1),(0,5)}};
klpList = {{ka_3},{ka_1,ka_2},{ka_1,ka_1,ka_1},{ka_2,la_1},{ka_1,ka_1,la_1}};
M = matrix table(klpList,gnList,
     (x,y) -> (sum(tempFactors(x,#y), z->product(#y, i->integral(y#i#0,y#i#1,z#i)))));
kernel M
```

The function `tempFactors` returns a list of how the factors of a monomial $\lambda_1\kappa_{a_1}\cdots\kappa_{a_k}$ can be distributed among components of a bouundary stratum. The code yields the output:

```
image | 0  -2304 |
      | 0  -1152 |
      | 0  -1    |
      | -1 24    |
      | 1  0     |
      | 0  96    |
      | 0  72    |
```

The kernel of M is incompatible with (17). Thus $X_1 - X_2$ is nonzero in $R^3(\mathcal{M}_{3,0}^{\lambda_2})$, and the pairing (14) is not perfect.

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